Table A.7-1 Summary of Differential Operations Involving the ∇-Operator in Cartesian Coordinates (x, y, z)

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{A}$$

$$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \tag{B}$$

$$\begin{aligned}(\boldsymbol{\tau}:\nabla\mathbf{v}) = {} & \tau_{xx}\left(\frac{\partial v_x}{\partial x}\right) + \tau_{xy}\left(\frac{\partial v_x}{\partial y}\right) + \tau_{xz}\left(\frac{\partial v_x}{\partial z}\right) \\ & + \tau_{yx}\left(\frac{\partial v_y}{\partial x}\right) + \tau_{yy}\left(\frac{\partial v_y}{\partial y}\right) + \tau_{yz}\left(\frac{\partial v_y}{\partial z}\right) \\ & + \tau_{zx}\left(\frac{\partial v_z}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v_z}{\partial y}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right)\end{aligned} \tag{C}$$

$$[\nabla s]_x = \frac{\partial s}{\partial x} \tag{D}$$

$$[\nabla s]_y = \frac{\partial s}{\partial y} \tag{E}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \tag{F}$$

$$[\nabla \times \mathbf{v}]_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \tag{G}$$

$$[\nabla \times \mathbf{v}]_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \tag{H}$$

$$[\nabla \times \mathbf{v}]_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{I}$$

$$[\nabla \cdot \boldsymbol{\tau}]_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{J}$$

$$[\nabla \cdot \boldsymbol{\tau}]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{K}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{L}$$

$$[\nabla^2 \mathbf{v}]_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \tag{M}$$

$$[\nabla^2 \mathbf{v}]_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \tag{N}$$

$$[\nabla^2 \mathbf{v}]_z = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \tag{O}$$

$$[\mathbf{v} \cdot\cdot \nabla\mathbf{w}]_x = v_x\left(\frac{\partial w_x}{\partial x}\right) + v_y\left(\frac{\partial w_x}{\partial y}\right) + v_z\left(\frac{\partial w_x}{\partial z}\right) \tag{P}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_y = v_x\left(\frac{\partial w_y}{\partial x}\right) + v_y\left(\frac{\partial w_y}{\partial y}\right) + v_z\left(\frac{\partial w_y}{\partial z}\right) \tag{Q}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_z = v_x\left(\frac{\partial w_z}{\partial x}\right) + v_y\left(\frac{\partial w_z}{\partial y}\right) + v_z\left(\frac{\partial w_z}{\partial z}\right) \tag{R}$$

$$\{\nabla \mathbf{v}\}_{xx} = \frac{\partial v_x}{\partial x} \tag{S}$$

$$\{\nabla \mathbf{v}\}_{xy} = \frac{\partial v_y}{\partial x} \tag{T}$$

$$\{\nabla \mathbf{v}\}_{xz} = \frac{\partial v_z}{\partial x} \tag{U}$$

$$\{\nabla \mathbf{v}\}_{yx} = \frac{\partial v_x}{\partial y} \tag{V}$$

$$\{\nabla \mathbf{v}\}_{yy} = \frac{\partial v_y}{\partial y} \tag{W}$$

$$\{\nabla \mathbf{v}\}_{yz} = \frac{\partial v_z}{\partial y} \tag{X}$$

$$\{\nabla \mathbf{v}\}_{zx} = \frac{\partial v_x}{\partial z} \tag{Y}$$

$$\{\nabla \mathbf{v}\}_{zy} = \frac{\partial v_y}{\partial z} \tag{Z}$$

$$\{\nabla \mathbf{v}\}_{zz} = \frac{\partial v_z}{\partial z} \tag{AA}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{xx} = (\mathbf{v} \cdot \nabla)\tau_{xx} \tag{BB}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{xy} = (\mathbf{v} \cdot \nabla)\tau_{xy} \tag{CC}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{xz} = (\mathbf{v} \cdot \nabla)\tau_{xz} \tag{DD}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{yx} = (\mathbf{v} \cdot \nabla)\tau_{yx} \tag{EE}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{yy} = (\mathbf{v} \cdot \nabla)\tau_{yy} \tag{FF}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{yz} = (\mathbf{v} \cdot \nabla)\tau_{yz} \tag{GG}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{zx} = (\mathbf{v} \cdot \nabla)\tau_{zx} \tag{HH}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{zy} = (\mathbf{v} \cdot \nabla)\tau_{zy} \tag{II}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{zz} = (\mathbf{v} \cdot \nabla)\tau_{zz} \tag{JJ}$$

where the operator $(\mathbf{v} \cdot \nabla) = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$

Table A.7-2 Summary of Differential Operations Involving the ∇-Operator in Cylindrical Coordinates (r, θ, z)

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \tag{A}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \tag{B}$$

$$\begin{aligned}(\boldsymbol{\tau}:\nabla\mathbf{v}) = {} & \tau_{rr}\left(\frac{\partial v_r}{\partial r}\right) + \tau_{r\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \tau_{rz}\left(\frac{\partial v_r}{\partial z}\right) \\ & + \tau_{\theta r}\left(\frac{\partial v_\theta}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \tau_{\theta z}\left(\frac{\partial v_\theta}{\partial z}\right) \\ & + \tau_{zr}\left(\frac{\partial v_z}{\partial r}\right) + \tau_{z\theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right)\end{aligned} \tag{C}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r} \tag{D}$$

$$[\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta} \tag{E}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \tag{F}$$

$$[\nabla \times \mathbf{v}]_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \tag{G}$$

$$[\nabla \times \mathbf{v}]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \tag{H}$$

$$[\nabla \times \mathbf{v}]_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \tag{I}$$

$$[\nabla \cdot \boldsymbol{\tau}]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r} \tag{J}$$

$$[\nabla \cdot \boldsymbol{\tau}]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} \tag{K}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz} \tag{L}$$

$$[\nabla^2 \mathbf{v}]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \tag{M}$$

$$[\nabla^2 \mathbf{v}]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \tag{N}$$

$$[\nabla^2 \mathbf{v}]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \tag{O}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_r = v_r\left(\frac{\partial w_r}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + v_z\left(\frac{\partial w_r}{\partial z}\right) \tag{P}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_\theta = v_r\left(\frac{\partial w_\theta}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + v_z\left(\frac{\partial w_\theta}{\partial z}\right) \tag{Q}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_z = v_r\left(\frac{\partial w_z}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_z}{\partial \theta}\right) + v_z\left(\frac{\partial w_z}{\partial z}\right) \tag{R}$$

$$\{\nabla \mathbf{v}\}_{rr} = \frac{\partial v_r}{\partial r} \tag{S}$$

$$\{\nabla \mathbf{v}\}_{r\theta} = \frac{\partial v_\theta}{\partial r} \tag{T}$$

$$\{\nabla \mathbf{v}\}_{rz} = \frac{\partial v_z}{\partial r} \tag{U}$$

$$\{\nabla \mathbf{v}\}_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \tag{V}$$

$$\{\nabla \mathbf{v}\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \tag{W}$$

$$\{\nabla \mathbf{v}\}_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} \tag{X}$$

$$\{\nabla \mathbf{v}\}_{zr} = \frac{\partial v_r}{\partial z} \tag{Y}$$

$$\{\nabla \mathbf{v}\}_{z\theta} = \frac{\partial v_\theta}{\partial z} \tag{Z}$$

$$\{\nabla \mathbf{v}\}_{zz} = \frac{\partial v_z}{\partial z} \tag{AA}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{rr} = (\mathbf{v}\cdot\nabla)\tau_{rr} - \frac{v_\theta}{r}(\tau_{r\theta} + \tau_{\theta r}) \tag{BB}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{r\theta} = (\mathbf{v}\cdot\nabla)\tau_{r\theta} + \frac{v_\theta}{r}(\tau_{rr} - \tau_{\theta\theta}) \tag{CC}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{rz} = (\mathbf{v}\cdot\nabla)\tau_{rz} - \frac{v_\theta}{r}\tau_{\theta z} \tag{DD}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta r} = (\mathbf{v}\cdot\nabla)\tau_{\theta r} + \frac{v_\theta}{r}(\tau_{rr} - \tau_{\theta\theta}) \tag{EE}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta\theta} = (\mathbf{v}\cdot\nabla)\tau_{\theta\theta} + \frac{v_\theta}{r}(\tau_{r\theta} + \tau_{\theta r}) \tag{FF}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta z} = (\mathbf{v}\cdot\nabla)\tau_{\theta z} + \frac{v_\theta}{r}\tau_{rz} \tag{GG}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{zr} = (\mathbf{v}\cdot\nabla)\tau_{zr} - \frac{v_\theta}{r}\tau_{z\theta} \tag{HH}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{z\theta} = (\mathbf{v}\cdot\nabla)\tau_{z\theta} + \frac{v_\theta}{r}\tau_{zr} \tag{II}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{zz} = (\mathbf{v}\cdot\nabla)\tau_{zz} \tag{JJ}$$

where the operator $(\mathbf{v}\cdot\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}$

Table A.7-3 Summary of Differential Operations Involving the ∇-Operator in Spherical Coordinates (r, θ, ϕ)

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \theta}(v_\theta\ \sin\ \theta) + \frac{1}{r\ \sin\ \theta}\frac{\partial v_\phi}{\partial \phi} \tag{A}$$

$$(\nabla^2 s) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial s}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 s}{\partial \phi^2} \tag{B}$$

$$\begin{aligned}(\boldsymbol{\tau}:\nabla\mathbf{v}) = {} & \tau_{rr}\left(\frac{\partial v_r}{\partial r}\right) + \tau_{r\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \tau_{r\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r}\right) \\ & + \tau_{\theta r}\left(\frac{\partial v_\theta}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \tau_{\theta\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r}\cot\theta\right) \\ & + \tau_{\phi r}\left(\frac{\partial v_\phi}{\partial r}\right) + \tau_{\phi\theta}\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \theta}\right) + \tau_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta\right)\end{aligned} \tag{C}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r} \tag{D}$$

$$[\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta} \tag{E}$$

$$[\nabla s]_\phi = \frac{1}{r\ \sin\ \theta}\frac{\partial s}{\partial \phi} \tag{F}$$

$$[\nabla \times \mathbf{v}]_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \tag{G}$$

$$[\nabla \times \mathbf{v}]_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi) \tag{H}$$

$$[\nabla \times \mathbf{v}]_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \tag{I}$$

$$[\nabla \cdot \boldsymbol{\tau}]_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \theta}(\tau_{\theta r}\ \sin\ \theta) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \phi}\tau_{\phi r} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \tag{J}$$

$$[\nabla \cdot \boldsymbol{\tau}]_\theta = \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\theta}) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\ \sin\ \theta) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \phi}\tau_{\phi\theta} + \frac{(\tau_{\theta r} - \tau_{r\theta}) - \tau_{\phi\phi}\ \cot\ \theta}{r} \tag{K}$$

$$[\nabla \cdot \boldsymbol{\tau}]_\phi = \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\phi}) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \theta}(\tau_{\theta\phi}\ \sin\ \theta) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial \phi}\tau_{\phi\phi} + \frac{(\tau_{\phi r} - \tau_{r\phi}) + \tau_{\phi\theta}\ \cot\ \theta}{r} \tag{L}$$

$$[\nabla^2 \mathbf{v}]_r = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)\right) + \frac{1}{r^2\ \sin\ \theta}\frac{\partial}{\partial \theta}\left(\sin\ \theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\ \sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2\ \sin\ \theta}\frac{\partial}{\partial \theta}(v_\theta\ \sin\ \theta) - \frac{2}{r^2\ \sin\ \theta}\frac{\partial v_\phi}{\partial \phi} \tag{M}$$

$$[\nabla^2 \mathbf{v}]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\ \theta}\frac{\partial}{\partial \theta}(v_\theta\ \sin\ \theta)\right) + \frac{1}{r^2\ \sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\ \cot\ \theta}{r^2\ \sin\ \theta}\frac{\partial v_\phi}{\partial \phi} \tag{N}$$

$$[\nabla^2 \mathbf{v}]_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\ \theta}\frac{\partial}{\partial \theta}(v_\phi\ \sin\ \theta)\right) + \frac{1}{r^2\ \sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\ \sin\ \theta}\frac{\partial v_r}{\partial \phi} + \frac{2\ \cot\ \theta}{r^2\ \sin\ \theta}\frac{\partial v_\theta}{\partial \phi} \tag{O}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_r = v_r\left(\frac{\partial w_r}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + v_\phi\left(\frac{1}{r\ \sin\ \theta}\frac{\partial w_r}{\partial \phi} - \frac{w_\phi}{r}\right) \tag{P}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_\theta = v_r\left(\frac{\partial w_\theta}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + v_\phi\left(\frac{1}{r\ \sin\ \theta}\frac{\partial w_\theta}{\partial \phi} - \frac{w_\phi}{r}\cot\ \theta\right) \tag{Q}$$

$$[\mathbf{v} \cdot \nabla\mathbf{w}]_\phi = v_r\left(\frac{\partial w_\phi}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\phi}{\partial \theta}\right) + v_\phi\left(\frac{1}{r\ \sin\ \theta}\frac{\partial w_\phi}{\partial \phi} + \frac{w_r}{r} + \frac{w_\theta}{r}\cot\ \theta\right) \tag{R}$$

$$\{\nabla \mathbf{v}\}_{rr} = \frac{\partial v_r}{\partial r} \tag{S}$$

$$\{\nabla \mathbf{v}\}_{r\theta} = \frac{\partial v_\theta}{\partial r} \tag{T}$$

$$\{\nabla \mathbf{v}\}_{r\phi} = \frac{\partial v_\phi}{\partial r} \tag{U}$$

$$\{\nabla \mathbf{v}\}_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \tag{V}$$

$$\{\nabla \mathbf{v}\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \tag{W}$$

$$\{\nabla \mathbf{v}\}_{\theta\phi} = \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} \tag{X}$$

$$\{\nabla \mathbf{v}\}_{\phi r} = \frac{1}{r \sin \theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \tag{Y}$$

$$\{\nabla \mathbf{v}\}_{\phi\theta} = \frac{1}{r \sin \theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r}\cot\theta \tag{Z}$$

$$\{\nabla \mathbf{v}\}_{\phi\phi} = \frac{1}{r \sin \theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta \tag{AA}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{rr} = (\mathbf{v}\cdot\nabla)\tau_{rr} - \left(\frac{v_\theta}{r}\right)(\tau_{r\theta} + \tau_{\theta r}) - \left(\frac{v_\phi}{r}\right)(\tau_{r\phi} + \tau_{\phi r}) \tag{BB}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{r\theta} = (\mathbf{v}\cdot\nabla)\tau_{r\theta} + \left(\frac{v_\theta}{r}\right)(\tau_{rr} - \tau_{\theta\theta}) - \left(\frac{v_\phi}{r}\right)(\tau_{\phi\theta} + \tau_{r\phi}\cot\theta) \tag{CC}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{r\phi} = (\mathbf{v}\cdot\nabla)\tau_{r\phi} - \left(\frac{v_\theta}{r}\right)\tau_{\theta\phi} + \left(\frac{v_\phi}{r}\right)[(\tau_{rr} - \tau_{\phi\phi}) + \tau_{r\theta}\cot\theta] \tag{DD}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta r} = (\mathbf{v}\cdot\nabla)\tau_{\theta r} + \left(\frac{v_\theta}{r}\right)(\tau_{rr} - \tau_{\theta\theta}) - \left(\frac{v_\phi}{r}\right)(\tau_{\theta\phi} + \tau_{\phi r}\cot\theta) \tag{EE}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta\theta} = (\mathbf{v}\cdot\nabla)\tau_{\theta\theta} + \left(\frac{v_\theta}{r}\right)(\tau_{r\theta} + \tau_{\theta r}) - \left(\frac{v_\phi}{r}\right)(\tau_{\theta\phi} + \tau_{\phi\theta})\cot\theta \tag{FF}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta\phi} = (\mathbf{v}\cdot\nabla)\tau_{\theta\phi} + \left(\frac{v_\theta}{r}\right)\tau_{r\phi} + \left(\frac{v_\phi}{r}\right)[\tau_{\theta r} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta] \tag{GG}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\phi r} = (\mathbf{v}\cdot\nabla)\tau_{\phi r} - \left(\frac{v_\theta}{r}\right)\tau_{\phi\theta} + \left(\frac{v_\phi}{r}\right)[(\tau_{rr} - \tau_{\phi\phi}) + \tau_{\theta r}\cot\theta] \tag{HH}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\phi\theta} = (\mathbf{v}\cdot\nabla)\tau_{\phi\theta} + \left(\frac{v_\theta}{r}\right)\tau_{\phi r} + \left(\frac{v_\phi}{r}\right)[\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta] \tag{II}$$

$$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\phi\phi} = (\mathbf{v}\cdot\nabla)\tau_{\phi\phi} + \left(\frac{v_\phi}{r}\right)[(\tau_{r\phi} + \tau_{\phi r}) + (\tau_{\theta\phi} + \tau_{\phi\theta})\cot\theta] \tag{JJ}$$

where the operator $(\mathbf{v}\cdot\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$