Mathematical Preliminaries

We begin the course with a review of vectors, & tensors, and associated calculus, as they form natural language of transport phenomena.

Scalars

A scalar is a quantity that has a ***magnitude only***. e.g. density, temperature.

Vectors

A vector is defined as a quantity of a given ***magnitude and a direction***. e.g. velocity, $\mathbf{v}$. We will denote the magnitude of a vector by $|\mathbf{v}|$ or simply v.

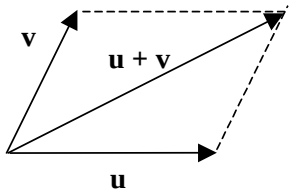
Two vectors **u** and **v** are said to be equal if their magnitudes are equal and when they point in the same direction; *they do not have to be collinear or have the same point of origin.*

Gibbs Notation:	$\mathbf{v}$	
Cartesian or index notation:	v_i	i=1, 2 or 3

In terms of unit vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ in the coordinate directions, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$

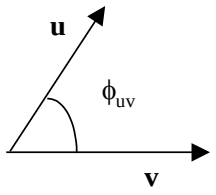
Vector addition

Gibbs notation:	$\mathbf{u} + \mathbf{v}$
Cartesian or index notation	$\{u_i + v_i\}$



Vector Multiplication

scalar product: consider two vectors oriented with an angle ϕ_{uv} to each other. A scalar product $\mathbf{u}\cdot\mathbf{v}$ is defined by $\mathbf{u}\cdot\mathbf{v} = uv\cos\phi_{uv}$.



It then follows that $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$ where δ_{ij} is the *Kronecker's delta*,

$$\text{i .e. } \delta_{ij} = \begin{cases} 1 \ \textit{if} \ i = j \\ 0 \ \textit{if} \ i \neq j \end{cases}.$$

Consequently, in component form $\mathbf{u}\cdot\mathbf{v} = \sum_i u_i v_i$

Summation convention: Sometimes, the summation over index is implicitly understood, if an index repeats itself. This is referred to as Einstein summation convention

$\mathbf{u}\cdot\mathbf{v} = u_i v_i \qquad \mathbf{u}\cdot\mathbf{u} = \sum u_i^2$

Scalar product is:

commutative:	$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
Not associative:	$(\mathbf{u}\cdot\mathbf{v})\mathbf{w} \neq \mathbf{u}(\mathbf{v}\cdot\mathbf{w})$
Distributive:	$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

The meaning of a vector should not depend on the coordinate system. Suppose we consider two coordinate systems $\{1, 2, 3\}$ and $\{1', 2', 3'\}$, then a given vector **u** can be expressed as

$$\mathbf{u} = \sum u_i \mathbf{e}_i = \sum u_i' \mathbf{e}_i'$$

As we can write

$$\mathbf{e}_i' = \sum_{j=1}^{3} (\mathbf{e}_i'\cdot\mathbf{e}_j)\mathbf{e}_j$$

we get

$$u_i = \sum_{j=1}^{3} u'_j (\mathbf{e}'_j \cdot \mathbf{e}_i), \; i = 1, \; 3$$

This can be written as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 & \mathbf{e}_1 \cdot \mathbf{e}'_3 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 & \mathbf{e}_2 \cdot \mathbf{e}'_3 \\ \mathbf{e}_3 \cdot \mathbf{e}'_1 & \mathbf{e}_3 \cdot \mathbf{e}'_2 & \mathbf{e}_3 \cdot \mathbf{e}'_3 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix}$$

In matrix language, we write this as

$$\underset{\sim}{u} = \underset{\approx}{Q}\, \underset{\sim}{u}'$$

Then, $\mathbf{u} \cdot \mathbf{u} = \underset{\sim}{u}^T \underset{\sim}{u} = \left(\underset{\approx}{Q} \underset{\sim}{u}' \right)^T \underset{\approx}{Q} \underset{\sim}{u}' = \underset{\sim}{u}'^T \underset{\approx}{Q}^{\mathrm{T}} \underset{\approx}{Q} \underset{\sim}{u}'$. But $\mathbf{u} \cdot \mathbf{u} = \mathbf{u}' \cdot \mathbf{u}' = \underset{\sim}{u}'^T \underset{\sim}{u}'$. Thus, we find that

$$\underset{\sim}{u}'^T \underset{\approx}{Q}^{\mathrm{T}} \underset{\approx}{Q} \underset{\sim}{u}' = \underset{\sim}{u}'^T \underset{\sim}{u}'$$

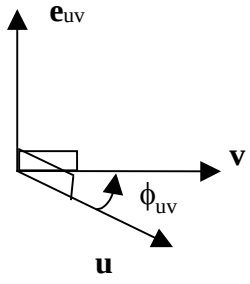
This can be true only if

$$\underset{\approx}{Q}^{\mathrm{T}} \underset{\approx}{Q} = \underset{\approx}{\delta}, \text{ so that } \underset{\approx}{Q}^{\mathrm{T}} = \underset{\approx}{Q}^{-1}$$

Here, $\underset{\approx}{\delta}$ is the (unit) identity matrix.

Vector Product (Cross Product)

A vector product of **u** and **v** is defined as $\mathbf{u} \times \mathbf{v} = uv \sin \phi_{uv} \mathbf{e}_{uv}$ where $\mathbf{e}_{uv}$ is a unit vector that is perpendicular to the plane formed by **u** and **v**. The direction of $\mathbf{e}_{uv}$ is such that (**u**, **v**, $\mathbf{e}_{uv}$) form a right-handed set. Thus $|\mathbf{u} \times \mathbf{v}|$ is the area of the parallelogram defined by **u** and **v**.



Consider a vector product involving unit vectors: $\mathbf{e}_i \times \mathbf{e}_j = \sum_k E_{ijk} \mathbf{e}_k$

where E_{ijk} is the permutation symbol given by

$$E_{ijk} = \begin{cases} 0 & i = j \text{ or } j = k \text{ or } k = i \\ 1 & ijk = 123, 231 \text{ or } 312 \\ -1 & ijk = 132, 213 \text{ or } 321 \end{cases}$$

Note that $E_{ijk} = E_{jki} = E_{kij}$.

Notation:

Gibbs notation: $\mathbf{u} \times \mathbf{v}$

Index $E_{ijk} u_j v_k$ (summation implied)

$$\mathbf{u} \times \mathbf{v} = \left(\sum u_j \mathbf{e}_j \right) \times \left(\sum v_k \mathbf{e}_k \right)$$

$$\therefore \mathbf{u} \times \mathbf{v} = \sum_{i,j,k} E_{ijk} u_j v_k \mathbf{e}_i$$

$$\therefore \mathbf{u} \times \mathbf{v} = \left(u_2 v_3 - u_3 v_2 \right) \mathbf{e}_1 + \left(u_3 v_1 - u_1 v_3 \right) \mathbf{e}_2 + \left(u_1 v_2 - u_2 v_1 \right) \mathbf{e}_3$$

Vector products are:

not commutative: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

not associative: $\mathbf{u} \times \left[\mathbf{v} \times \mathbf{w} \right] \neq \left[\mathbf{u} \times \mathbf{v} \right] \times \mathbf{w}$

distributive: $\left(\mathbf{u} + \mathbf{v} \right) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

Multiple Products of Vectors

Consider the following:

(i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \sum_{i,j,k} a_i E_{ijk} b_j c_k = \sum_j b_j \left(\sum_{i,k} E_{jki} c_k a_i \right) = \mathbf{b} \cdot \left(\mathbf{c} \times \mathbf{a} \right)$

Proceeding in this manner, we can show

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

(ii) Two identities involving the permutation symbols can be verified.

$$\sum_k E_{ijk} E_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

$$\sum_j \sum_k E_{ijk} E_{ljk} = 2 \delta_{il}$$

With the help of these one can prove several relations involving multiple vector products.

(iii) consider $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\left(\sum_i a_i\mathbf{e}_i\right)\times\left[\sum_j b_j\mathbf{e}_j\times\sum_k c_k\mathbf{e}_k\right]=\left(\sum_i a_i\mathbf{e}_i\right)\times\sum_{ljk}E_{ljk}b_jc_k\mathbf{e}_l$$

$$=\sum_{miljk}E_{mil}E_{ljk}a_ib_jc_k\mathbf{e}_m=\sum_{ijkm}\left\{\left(\sum_l E_{mil}E_{jkl}\right)a_ib_jc_k\right\}\mathbf{e}_m$$

$$=\sum_{ijkm}\left(\delta_{jm}\delta_{ki}-\delta_{ji}\delta_{km}\right)a_ib_jc_k\mathbf{e}_m=\sum_{mi}a_ib_mc_i\mathbf{e}_m-\sum_{mi}a_ib_ic_m\mathbf{e}_m$$

$$=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

Thus we have

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

(iv) It readily follows from the above that

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{c}\cdot\mathbf{a})\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\mathbf{a}$$

$$=-\left[(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}\right]=-\mathbf{c}\times(a\times b)$$

(v) We can also show that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

The left hand side can be written as

$$\left(\sum_{ijk}E_{ijk}a_jb_k\mathbf{e}_i\right)\cdot\left(\sum_{mnp}E_{mnp}c_nd_p\mathbf{e}_m\right)=\sum_{ijknp}E_{ijk}E_{inp}a_jb_kc_nd_p$$

$$=\sum_{jknp}\left(\delta_{jn}\delta_{kp}-\delta_{jp}\delta_{kn}\right)a_jb_kc_nd_p$$

$$=\sum_{jk}\left(a_jc_jb_kd_k-a_jd_jb_kc_k\right)$$

$$=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Second Order Tensor

A second order tensor in 3-D space is an entity that can be represented as an ordered set of 9 numbers, each of which is associated with two directions.

$$\boldsymbol{\tau} = \sum_{ij} \tau_{ij} \mathbf{e}_i \mathbf{e}_j = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

Again, it is understood from "i,j" under the summation sign that the summation is over i = 1,3, and j = 1,3. The second equality merely illustrates another way of visualizing the tensor. It is clear that the coefficients τ_{ij} can be organized in the form of a square matrix, and matrix algebra can be used to handle manipulations, as illustrated later in this section.

$\boldsymbol{\tau}^t$ is defined as transpose of $\boldsymbol{\tau}$.

$$\boldsymbol{\tau}^t = \sum_{ij} \tau_{ji} \mathbf{e}_i \mathbf{e}_j$$

$\boldsymbol{\tau} = \boldsymbol{\tau}^t \Rightarrow$ symmetric tensor

$\boldsymbol{\tau} = -\boldsymbol{\tau}^t \Rightarrow$ antisymmetric tensor

Any tensor can be partitioned into symmetric and antisymmetric parts

$$\boldsymbol{\tau} = \boldsymbol{\tau}^S + \boldsymbol{\tau}^A,$$

where

$$\boldsymbol{\tau}^S = \frac{1}{2}\left(\boldsymbol{\tau} + \boldsymbol{\tau}^t\right); \boldsymbol{\tau}^A = \frac{1}{2}\left(\boldsymbol{\tau} - \boldsymbol{\tau}^t\right)$$

What does uv (where u and v are vectors) mean? This is referred to as a **dyad**. When the two vectors involved in the dyad are unit vectors of an orthogonal coordinate system (i.e. $\mathbf{u} = \mathbf{e_i}$ and $\mathbf{v} = \mathbf{e_j}$) we'll refer to them as a **unit dyad**. In 3D, these are 9 unit dyads. The order of the vector directions is significant and so $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_1$ mean entirely different unit dyads.

From your earlier mathematics courses dealing with vectors and matrices, you know that a vector is nothing but a sequence of numbers arranged in some particular order; unless someone tells you what that sequence refers to, one cannot assign any physical meaning. For example, when we refer to position vectors, the three elements of the vector refer to the coordinate locations in some specific orthogonal coordinate systems. If it were a velocity vector, the three elements refer to velocity components in the three coordinate directions. Similarly, the matrix elements have no meaning per se, unless someone tells you how to read the entries. In a similar manner, a tensor has no specific physical

meaning other than a logical means of organizing a set of (nine) numbers. Some one will have to tell us about the logical organization of the numbers to form a tensor before we can physically interpret it. Even though this sounds rather abstract when you first encounter it, we will see in this course that tensors arise naturally in force balances.

Just like vectors, a tensor should not change with coordinate systems, although its component will change. So if we consider two coordinate systems $\{e_i\}$ and $\{e'_i\}$,

$$\boldsymbol{\tau} = \sum_{ij} \tau_{ij} \mathbf{e}_i \mathbf{e}_j = \sum_{ij} \tau'_{ij} \mathbf{e}'_i \mathbf{e}'_j$$

In matrix form, this is written as

$$\underset{\sim}{e}^T \underset{\sim}{\underset{\sim}{\tau}} \underset{\sim}{e} = \underset{\sim}{e}'^T \underset{\sim}{\underset{\sim}{\tau}}' \underset{\sim}{e}' \qquad \text{(a)}$$

How are the matrices $(\underset{\sim}{\underset{\sim}{\tau}})$ + $(\underset{\sim}{\underset{\sim}{\tau}}')$ related? Recall that

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 & \mathbf{e}_1 \cdot \mathbf{e}'_3 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 & \mathbf{e}_2 \cdot \mathbf{e}'_3 \\ \mathbf{e}_3 \cdot \mathbf{e}'_1 & \mathbf{e}_3 \cdot \mathbf{e}'_2 & \mathbf{e}_3 \cdot \mathbf{e}'_3 \end{bmatrix}}_{\underset{\sim}{Q}} \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix}$$

In matrix form, this is written as $\underset{\sim}{e} = \underset{\sim}{Q}\underset{\sim}{e}'$, so $\underset{\sim}{e}^T \underset{\sim}{\underset{\sim}{\tau}} \underset{\sim}{e} = \underset{\sim}{e}'^T \underset{\sim}{Q}^T \underset{\sim}{\underset{\sim}{\tau}} \underset{\sim}{Q} \underset{\sim}{e}'$.

Comparing this with equation (a) above, we conclude that

$$\underset{\sim}{\underset{\sim}{\tau}}' = \underset{\sim}{Q}^T \underset{\sim}{\underset{\sim}{\tau}} \underset{\sim}{Q}$$

Tensor Manipulations

(1)

$$\begin{aligned} \mathbf{u} \cdot \mathbf{T} &= \left(\sum_{\mathbf{i}} u_i \mathbf{e}_i \right) \cdot \left(\sum_{k,j} T_{kj} \mathbf{e}_k \mathbf{e}_j \right) \\ &= \sum_{i,k,j} u_i T_{kj} \left(\mathbf{e}_i \cdot \mathbf{e}_k \right) \mathbf{e}_j = \sum_{i,k,j} u_i T_{kj} \left(\delta_{ik} \right) \mathbf{e}_j \\ &= \sum_{i,j} u_i T_{ij} \mathbf{e}_j \text{ (as } \delta_{ik} \text{is zero unless i = k and one when i = k)} \end{aligned}$$

(2)

$$\mathbf{T} \cdot \mathbf{u} = \sum_{i,j} T_{ij} u_j \mathbf{e}_i = \mathbf{u} \cdot \mathbf{T}^t$$

$$\mathbf{u}\cdot\mathbf{T}\cdot\mathbf{v}=\sum_{i,j}u_i T_{ij} v_j$$

$$\mathbf{S}\cdot\mathbf{T}=\sum_{i,j,k}S_{ij}T_{jk}\mathbf{e}_i\mathbf{e}_k$$

(3) Scalar product of (2^{nd} order) tensors, also referred to as a dyadic product:

$$\underbrace{\mathbf{S}:\mathbf{T}}_{\text{Gibbs}}=\left(\sum_{i,j}S_{ij}\mathbf{e}_i\mathbf{e}_j\right):\left(\sum_{k,l}T_{kl}\mathbf{e}_k\mathbf{e}_l\right)$$

$$=\sum_{i,j,k,l}S_{ij}T_{kl}\mathbf{e}_i\mathbf{e}_j:\mathbf{e}_k\mathbf{e}_l$$

$$=\sum_{i,j,k,l}S_{ij}T_{kl}\delta_{jk}\left(\mathbf{e}_i\cdot\mathbf{e}_l\right)$$ (where we have first contracted the inner pair of vectors)

$$=\sum_{i,j,k,l}S_{ij}T_{kl}\delta_{jk}\delta_{il}$$

$$=\sum_{i,j}S_{ij}T_{ji}$$

(4) $$\mathbf{u}\times\mathbf{T}=\left(\sum_i u_i\mathbf{e}_i\right)\times\sum_{jk}T_{jk}\mathbf{e}_j\mathbf{e}_k \quad =\sum_{mijk}E_{mij}u_i T_{jk}\ \mathbf{e}_m\mathbf{e}_k$$

Invariant Quantities

In fluid mechanics, we'll frequently be interested in quantities that are independent of the coordinate system, i.e. quantities that will not change upon translation or rotation of the coordinate axes. All scalar quantities automatically satisfy this. Consider a vector **u**. In two different coordinate systems $\{\mathbf{e_i}\}$ and $\{\mathbf{e}'_i\}$, it can be written as

$$\mathbf{u} = \Sigma\, u_i \mathbf{e_i} = \Sigma\, u'_i \mathbf{e}'_i \ ,$$

and clearly u_i and u'_i are dependent on choice of coordinate systems. However, we know that $\sqrt{\mathbf{u}\cdot\mathbf{u}} = u \ \left(\equiv |\mathbf{u}|\right)$ is independent of coordinate system. Thus, u is the (only) invariant in a vector.

A second order tensor has three invariants. For a tensor τ, from properties of matrices, we can conclude that three independent scalars can be formed by taking trace of $\boldsymbol{\tau}$, $\boldsymbol{\tau}^2$ and $\boldsymbol{\tau}^3$. These scalars I, II, & III are invariant under coordinate transformation

$$\begin{aligned} I &= \mathrm{tr}\,(\boldsymbol{\tau}) = \sum_i \tau_{ii} \\ II &= \mathrm{tr}(\boldsymbol{\tau})^2 = \sum_{ij} \tau_{ij}\tau_{ji} \\ III &= \mathrm{tr}(\boldsymbol{\tau})^3 = \sum_l \sum_j \sum_k \tau_{ij}\tau_{jk}\tau_{kl} \end{aligned}$$

The most commonly encountered invariants are I_1, I_2 and I_3, defined according to

$$I_1 = I, \qquad I_2 = \frac{1}{2}(I^2 - II), \qquad I_3 = \frac{1}{6}(I^3 - 3I \cdot II + 2III) = \det(\boldsymbol{\tau})$$

Where do these come from? Second order tensors, being square matrices, have eigenvalues associated with them. Later in the course, we'll discuss these in the context of principal stresses and principal rates of deformation. For now, it suffices to say that these eigenvalues are found by solving the characteristic equation

$$\det(\boldsymbol{\tau} - \lambda\mathbf{I}) = 0 \text{ (which in matrix notation takes the form } \det(\underset{\approx}{\tau} - \lambda\underset{\approx}{\delta}) = 0\,)$$

where **I** is an identity tensor defined as $\mathbf{I} = \sum_{ij} \delta_{ij}\mathbf{e}_i\mathbf{e}_j$, $\underset{\approx}{\delta}$ is the identity matrix, and λ is the eigenvalue. Expanding the characteristic equation (for 3-D), one gets

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

(and there are 3 eigenvalues).

Vector Differential Operators

In transport phenomena, we are frequently interested in field variables, which are functions of position; in particular, we wish to represent how these field variables vary spatially. Gradient operators naturally enter in the analysis of such problems. Let us first focus on a rectangular coordinate system so that the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are constants when integrating or differentiating.

Gradient operator, ∇, "del" $\qquad \nabla = \sum \mathbf{e}_i \frac{\partial}{\partial x_i}(\ )$

when this is applied to a scalar $f(\mathbf{x},t)$, $\qquad \nabla f = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \frac{\partial f}{\partial x_3}\mathbf{e}_3$

Thus, the gradient of a scalar is a vector. The value of f at a location $\delta\mathbf{x}$ away from $\mathbf{x}$, is given (to first order accuracy) by

$$f(\mathbf{x} + \delta\mathbf{x}, t) \approx f(\mathbf{x}, t) + \nabla f \cdot \delta\mathbf{x} + O(|\delta\mathbf{x}|^2).$$

When gradient operator is applied to a vector $\mathbf{u}(\mathbf{x}, t)$, we get a second order tensor.

$$\overbrace{\nabla\mathbf{u}}^{\text{Gibbs notation}} = \overbrace{\sum_{ij} \frac{\partial u_j}{\partial x_i}}^{\text{index notation}} \mathbf{e}_i\mathbf{e}_j = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Divergence of a vector field represents a scalar product of ∇ with the vector field.

$$\underbrace{\nabla \cdot \mathbf{u}}_{\text{Gibbs notation}} = \underbrace{\sum_i \frac{\partial u_i}{\partial x_i}}_{\text{index notation}}$$

Divergence of a vector is

not commutative: $\qquad \nabla \cdot \mathbf{u} \neq \mathbf{u} \cdot \nabla$

not associative: $\qquad \nabla \cdot s\mathbf{u} \neq \nabla s \cdot \mathbf{u}$

distributive: $\qquad \nabla \cdot (\mathbf{u} + \mathbf{v}) = \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{v}$

Divergence of a tensor, $\nabla \cdot \boldsymbol{\tau}$

$$\underbrace{\nabla \cdot \boldsymbol{\tau}}_{\text{Gibbs notation}} = \underbrace{\sum_{i,j} \frac{\partial \tau_{ij}}{\partial x_i} \mathbf{e}_j}_{\text{index notation}}$$

Curl of a vector field is the cross product of ∇ with a vector field **u**.

$$\underbrace{\nabla \times \mathbf{v}}_{\text{Gibbs notation}} = \underbrace{\sum_{ijk} E_{ijk} \frac{\partial v_k}{\partial x_j} \mathbf{e}_i}_{\text{index notation}}$$

$(\nabla \times \mathbf{v})$ is sometimes written as (curl **v**).

Laplacian of a scalar field, $\nabla \cdot \nabla \left(= \nabla^2 \right)$

$$\nabla^2 f = \nabla \cdot \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2},$$

so we write

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial x_i^2}(\ \ )$$

Laplacian of a vector field, **v**:

$$\nabla^2 \mathbf{v} = \nabla \cdot \left(\nabla \mathbf{v} \right) = \sum_{i,j} \frac{\partial^2 v_j}{\partial x_i^2} \mathbf{e}_j$$

Vector-tensor identities **(Solved examples in Deen, pp. 562-563)**

$$\nabla \cdot \left(\nabla \mathbf{v} + \left(\nabla \mathbf{v} \right)^t \right) = \nabla^2 \mathbf{v} + \nabla \left(\nabla \cdot \mathbf{v} \right)$$

$$\nabla \cdot \left(\boldsymbol{\tau} \cdot \mathbf{v} \right) = \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \left(\nabla \cdot \boldsymbol{\tau} \right)$$

These can be proved by expressing the terms in component form in a rectangular co-ordinate system.

Time Derivatives of a field variable

Consider a scalar field f(**x**,t). The Eulerian derivative of f is defined by $\left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}}$ and it describes observations recorded by an observer stationed at **x**.

Next suppose that the observer is at **x** at time t, and is traveling at a velocity, **c**. Then the rate of change of f that this observer will record is no longer the Eulerian derivative (unless **c** = **0**).

$$\left(\frac{df}{dt}\right) = \lim_{\delta t \to 0} \frac{f\left(\mathbf{x} + \mathbf{c}\delta t, t + \delta t\right) - f\left(\mathbf{x}, t\right)}{\delta t}$$
$$= \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} + \mathbf{c} \cdot \nabla f$$

$\frac{d(\cdot)}{dt}$ is thus a **convective derivative**.

In analysis of transport phenomena, we'll frequently be concerned with changes seen by an observer who is moving at the local velocity of the fluid itself. Denoting this velocity by **v**, we write

$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} + \mathbf{v} \cdot \nabla f$$

and $\frac{D}{Dt}$ is reserved to represent this type of derivative, known as **material derivative**.

Time derivatives of vector fields (**u**(**x**,t)) are defined similarly:

Eulerian Derivative: $\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}}$

Convective Derivative: $\frac{d\mathbf{u}}{dt} = \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}} + \mathbf{c} \cdot \nabla \mathbf{u}$

Material Derivative: $\frac{D\mathbf{u}}{Dt} = \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}} + \mathbf{v} \cdot \nabla \mathbf{u}$

Coordinate transformation

The definitions of vector differential operators in Cartesian component forms described earlier are limited to rectangular coordinate systems. We use in practice cylindrical and spherical coordinate systems quite frequently, and the gradient operators are more complex in these coordinate systems. It is, however, straightforward to derive appropriate definitions in these systems. To see how this is done, let us suppose we have a rectangular coordinate system (x_1, x_2, x_3) and another one (ξ_1, ξ_2, ξ_3) which is not rectangular (but is locally orthogonal).

If we have a field variable $f(x_1, x_2, x_3)$, this can be rewritten as $g(\xi_1, \xi_2, \xi_3)$. A different symbol 'g' is used to emphasize that the functional forms of f & g will, in general, be different. The chain rule for differentiation is written as:

$$\left(\frac{\partial f}{\partial x_1}\right)_{x_2,x_3} = \left(\frac{\partial g}{\partial \xi_1}\right)_{\xi_2,\xi_3}\left(\frac{\partial \xi_1}{\partial x_1}\right)_{x_2,x_3} + \left(\frac{\partial g}{\partial \xi_2}\right)_{\xi_1,\xi_3}\left(\frac{\partial \xi_2}{\partial x_1}\right)_{x_2,x_3} + \left(\frac{\partial g}{\partial \xi_3}\right)_{\xi_1,\xi_2}\left(\frac{\partial \xi_3}{\partial x_1}\right)_{x_2,x_3} \quad (1)$$

(and similarly for differentiation with respect to x_2 & x_3). Then if $\mathbf{e}'_1, \mathbf{e}'_2$ and $\mathbf{e}'_3$ denote unit vectors in the ξ_1, ξ_2, and ξ_3 directions at the point (ξ_1, ξ_2, ξ_3), then we can write

$$\mathbf{e}_i = \sum_j \left(\mathbf{e}_i \cdot \mathbf{e}'_j\right)\mathbf{e}'_j, \quad i = 1,3$$

so that

$$\nabla f = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\right)\mathbf{e}_i = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial g}{\partial \xi_j}\right)\left(\frac{\partial \xi_j}{\partial x_i}\right)\left(\mathbf{e}_i \cdot \mathbf{e}'_k\right)\mathbf{e}'_k$$

Example 1: Let (x, y, z) be a rectangular coordinate system. Let (r, θ, z) be a cylindrical coordinate system having the same origin as the rectangular coordinate system and let the z-axes of the two systems be collinear. Express ∇a and $\nabla \cdot \mathbf{v}$, where $\mathbf{v}$ is a vector field and a is a scalar quantity, in a cylindrical coordinate system.

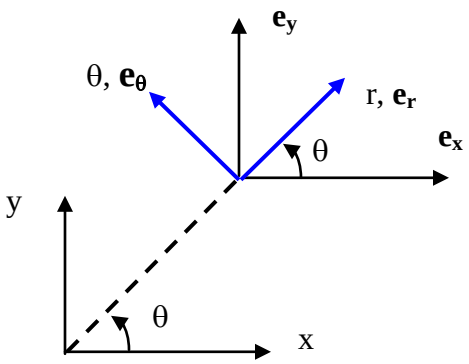
Clearly, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$, $z = z$ (see sketch in next page)

According to chain rule, equation (1) above :

$$\left.\begin{aligned} \frac{\partial(\ )}{\partial x} &= \cos\theta\frac{\partial(\ )}{\partial r} + \left(\frac{-\sin\theta}{r}\right)\frac{\partial(\ )}{\partial \theta} + 0\frac{\partial(\ )}{\partial z} \\ \frac{\partial(\ )}{\partial y} &= \sin\theta\frac{\partial(\ )}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial(\ )}{\partial \theta} + 0\frac{\partial(\ )}{\partial z} \\ \frac{\partial(\ )}{\partial z} &= \frac{\partial(\ )}{\partial z} \end{aligned}\right\} (i)$$

Also

$$\left.\begin{aligned}\mathbf{e}_r &= \cos\theta\cdot\mathbf{e}_x + \sin\theta\,\mathbf{e}_y + o\,\mathbf{e}_z\\ \mathbf{e}_\theta &= (-\sin\theta)\mathbf{e}_x + \cos\theta\,\mathbf{e}_y + o\,\mathbf{e}_z\\ \mathbf{e}_z &= \mathbf{e}_z\end{aligned}\right\} \qquad (ii)$$



This can be rewritten as

$$\left.\begin{aligned}\mathbf{e}_x &= \cos\theta\ \mathbf{e}_r - \sin\theta\ \mathbf{e}_\theta\\ \mathbf{e}_y &= \sin\theta\ \mathbf{e}_r + \cos\theta\ \mathbf{e}_\theta\\ \mathbf{e}_z &= \mathbf{e}_z\end{aligned}\right\} \qquad (iii)$$

As

$$\nabla = \frac{\partial(\ )}{\partial x}\mathbf{e}_x + \frac{\partial(\ )}{\partial y}\mathbf{e}_y + \frac{\partial(\ )}{\partial z}\mathbf{e}_z$$

z-axis is pointing out of the paper

we obtain from (i) & (iii) that

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}. \qquad (iv)$$

Therefore

$$\nabla a = \mathbf{e}_r\frac{\partial a}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial a}{\partial\theta} + \mathbf{e}_z\frac{\partial a}{\partial z} \qquad (v)$$

One can deduce from (ii) that

$$\frac{\partial}{\partial r}(\mathbf{e}_r) = 0 \qquad (\text{as } \mathbf{e}_x, \mathbf{e}_y \ \&\ \mathbf{e}_z \text{ are constants})$$

$$\frac{\partial}{\partial\theta}(\mathbf{e}_r) = -\sin\theta\ \mathbf{e}_x + \cos\theta\cdot\mathbf{e}_y = \mathbf{e}_\theta$$

$$\frac{\partial}{\partial z}(\mathbf{e}_r) = \frac{\partial}{\partial z}(\mathbf{e}_\theta) = \frac{\partial}{\partial z}(\mathbf{e}_z) = 0$$

$$\frac{\partial}{\partial r}(\mathbf{e}_\theta) = 0 = \frac{\partial\mathbf{e}_z}{\partial r} = \frac{\partial\mathbf{e}_z}{\partial\theta}$$

$$\frac{\partial}{\partial\theta}(\mathbf{e}_\theta) = -\cos\theta\ \mathbf{e}_x - \sin\theta\ \mathbf{e}_y = -\mathbf{e}_r$$

$\therefore \nabla\cdot\mathbf{v}$, where $\mathbf{v}$ is a vector field, can be written in a cylindrical coordinate system as

$$\nabla \cdot \mathbf{v} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right)$$
$$= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Example 2: Express $\nabla \mathbf{v}$ in a cylindrical coordinate system. Following the same procedure as in divergence operations, we can deduce that

$$\begin{aligned}
\nabla \mathbf{v} = & \frac{\partial v_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \frac{\partial v_\theta}{\partial r}\cdot\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial v_z}{\partial r}\mathbf{e}_r\mathbf{e}_z \\
& + \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right)\mathbf{e}_\theta\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\mathbf{e}_\theta\mathbf{e}_z \\
& + \frac{\partial v_r}{\partial z}\mathbf{e}_z\mathbf{e}_r + \frac{\partial v_\theta}{\partial z}\cdot\mathbf{e}_z\mathbf{e}_\theta + \frac{\partial v_z}{\partial z}\mathbf{e}_z\mathbf{e}_z
\end{aligned}$$

Example 3: Express $\nabla \cdot \boldsymbol{\tau}$ in a cylindrical coordinate system.

$$\begin{aligned}
\nabla \cdot \boldsymbol{\tau} = & \left\{ \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right\} \cdot \left\{ \sum_{r,\theta,z} \tau_{ij}\mathbf{e}_i\mathbf{e}_j \right\} \\
& = \frac{\partial \tau_{rr}}{\partial r}\cdot\mathbf{e}_r + \frac{\partial \tau_{r\theta}}{\partial r}\cdot\mathbf{e}_\theta + \frac{\partial \tau_{rz}}{\partial r}\mathbf{e}_z \\
& + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta}\cdot\mathbf{e}_r + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}\cdot\mathbf{e}_\theta + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}\mathbf{e}_z \\
& + \frac{\partial \tau_{zr}}{\partial z}\cdot\mathbf{e}_r + \frac{\partial \tau_{z\theta}}{\partial z}\cdot\mathbf{e}_\theta + \frac{\partial \tau_{zz}}{\partial z}\cdot\mathbf{e}_z \\
& + \frac{1}{r}\mathbf{e}_\theta \cdot \left\{ \begin{array}{l}
\tau_{rr}\frac{\partial}{\partial \theta}(\mathbf{e}_r\mathbf{e}_r) + \tau_{r\theta}\frac{\partial}{\partial \theta}(\mathbf{e}_r\mathbf{e}_\theta) + \tau_{rz}\frac{\partial}{\partial \theta}(\mathbf{e}_r\mathbf{e}_z) \\
+ \tau_{\theta r}\frac{\partial}{\partial \theta}(\mathbf{e}_\theta\mathbf{e}_r) + \tau_{\theta\theta}\frac{\partial}{\partial \theta}(\mathbf{e}_\theta\mathbf{e}_\theta) + \tau_{\theta z}\frac{\partial}{\partial \theta}(\mathbf{e}_\theta\mathbf{e}_z) \\
+ \tau_{zr}\frac{\partial}{\partial \theta}(\mathbf{e}_z\mathbf{e}_r) + \tau_{z\theta}\frac{\partial}{\partial \theta}(\mathbf{e}_z\mathbf{e}_\theta) + \tau_{zz}\frac{\partial}{\partial \theta}(\mathbf{e}_z\mathbf{e}_z)
\end{array} \right\}
\end{aligned}$$

The final group of terms leads to

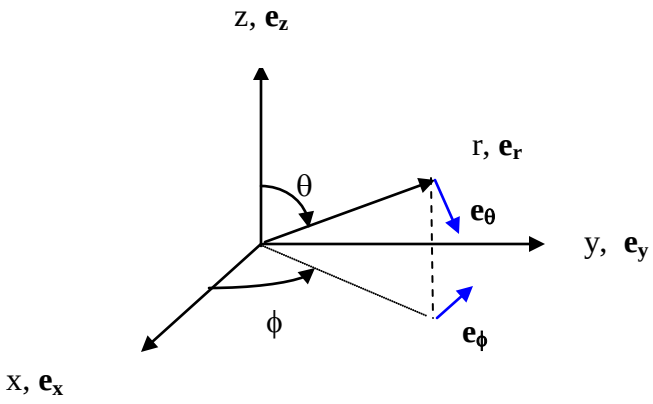
$$\frac{\tau_{rr}}{r}\mathbf{e}_r + \frac{\tau_{r\theta}}{r}\mathbf{e}_\theta + \frac{\tau_{rz}}{r}\mathbf{e}_z + \frac{\tau_{\theta r}}{r}\mathbf{e}_\theta - \frac{\tau_{\theta\theta}}{r}\mathbf{e}_r .$$

$$\therefore \quad \nabla\cdot\boldsymbol{\tau} = \left(\frac{\partial\tau_{rr}}{\partial r} + \frac{\tau_{rr}}{r} - \frac{\tau_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial z}\right)\mathbf{e}_r$$

$$+\left(\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{\tau_{r\theta}}{r} + \frac{\tau_{\theta r}}{r}\right)\mathbf{e}_\theta$$

$$+\left(\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r}\right)\mathbf{e}_z$$

Example 4:



Consider the Cartesian (x, y, z) and spherical (r, θ, ϕ) coordinate systems shown above.

Clearly,
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Furthermore,

$$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z$$
$$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{e}_x + \cos\theta\sin\phi\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z$$
$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y.$$

These can be inverted to obtain

$$\mathbf{e}_x = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi$$
$$\mathbf{e}_y = \sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi$$
$$\mathbf{e}_z = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta.$$

Show that $\nabla a = \mathbf{e}_r\frac{\partial a}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial a}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\phi\frac{\partial a}{\partial\phi}$, where a is a scalar.

Also show that, for any vector **v**,

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial\left(r^2 v_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(v_\theta\sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

Integral Transformations

In mathematical analysis of transport problems, the following integral theorems are very useful.

Gauss-Ostrogradskii Divergence Theorem

If V is a closed region in space enclosed by a surface S, then

$$\int_V (\nabla \cdot \mathbf{u}) dV = \int_S (\mathbf{n} \cdot \mathbf{u}) ds \qquad (a)$$

where **n** is the outwardly directed unit normal vector. Two closely allied theorems for scalars and tensors are:

$$\int_V \nabla f \; dV = \int_S \mathbf{n} f \; ds \qquad (b)$$

$$\int_V \nabla \cdot \boldsymbol{\tau} \, dV = \int_S (\mathbf{n} \cdot \boldsymbol{\tau}) ds \qquad (c)$$

The last relation "c" is also valid for a dyad **u v**. Similarly (b) is valid if f is replaced by a vector **u**.

Green's Identities

Suppose $\mathbf{u} = \phi \nabla \psi$ where ϕ and ψ are scalar functions with continuous 1st and 2nd derivatives in a closed region V. Then

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi$$

$\therefore$ from divergence theorem,

$$\int_V (\nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi) dV = \int_V \nabla \cdot \mathbf{u} dV = \int_S \mathbf{n} \cdot \mathbf{u} ds = \int \phi \frac{\partial \psi}{\partial n} ds \qquad (d)$$

where $\mathbf{n} \cdot \nabla \psi \equiv \frac{\partial \psi}{\partial n}$. This is known as **Green's first identity**.

Exchanging ϕ and ψ, we get

$$\int_V (\nabla\psi\cdot\nabla\phi+\psi\nabla^2\phi)dV=\int \psi\frac{\partial\phi}{\partial n}ds \qquad (e)$$

Combining (d) and (e), we get **Green's second identity**

$$\int_V (\phi\nabla^2\psi-\psi\nabla^2\phi)dV=\int_S (\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n})ds \qquad (f)$$

Leibniz Formula for Differentiating Integrals

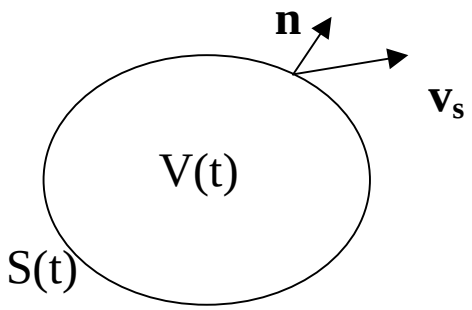
In transport problems, we frequently deal with differentiating expressions such as

$$G(t)=\int_{V(t)} f(\mathbf{x},t)d\mathbf{V}$$

where a field f is changing with time and the closed region of interest, V, is also changing with time. A generalization of Leibniz's rule in 1-D to multi-dimensional space takes the form

$$\frac{dG}{dt}=\int_{V(t)}\frac{\partial f}{\partial t}\cdot dV+\int_{S(t)}(\mathbf{n}\cdot\mathbf{v}_S)f\,ds$$

where the 2nd term accounts for the change in V. Here $\mathbf{v}_S$ denotes the velocity of the surface which can be a function of position and time and **n** is unit outward normal. Clearly, when the volume V is invariant with time, the 2nd term drops out.



If

$$\boldsymbol{b} = \int_{V(t)} \mathbf{a}(\mathbf{x}, t)\,dV$$

then

$$\frac{d\boldsymbol{b}}{dt} = \int_{V(t)} \frac{\partial \mathbf{a}}{\partial t}\,dV + \int_{S(t)} (n \cdot \mathbf{v}_S)\mathbf{a}\,ds$$

Position Vectors

A position vector, **r**, is a vector that extends from an arbitrary reference point (typically the origin of the coordinate system) to a point of interest in space.

$$\begin{aligned} \mathbf{r} &= r\,\mathbf{e}_r + z\,\mathbf{e}_z && \text{(in cylindrical)} \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 && \text{(in rectangular)} \end{aligned}$$

One can readily prove that

$$\begin{aligned} \nabla\mathbf{r} &= \delta && \text{(identity tensor)} \\ \nabla \cdot \mathbf{r} &= \Sigma\delta_{ii}\,[= 3 \quad \text{(in 3D)}] \\ \nabla \times \mathbf{r} &= \mathbf{0} \end{aligned}$$

By applying Gauss-Ostrogradskii theorem, one can show that

$$\int_{S_R} \mathbf{r}\,ds = \mathbf{0} \quad ; \quad \int_{S_R} \mathbf{rr}\,ds = \frac{4\pi R^4}{3}\delta$$

where S_R represents a spherical surface of radius R, having the same origin as used to define **r**.