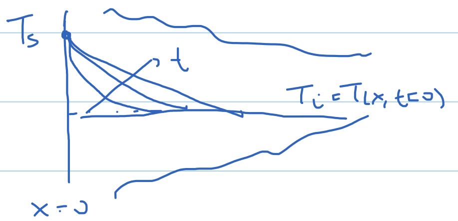
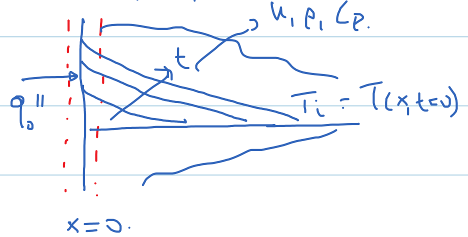
Wednesday, December 16, 2020
8:34 AM

$T(0,t) = T_s$



$q_0'' = -k \frac{\partial T}{\partial x}\Big|_{x=0}$



$$\frac{T - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

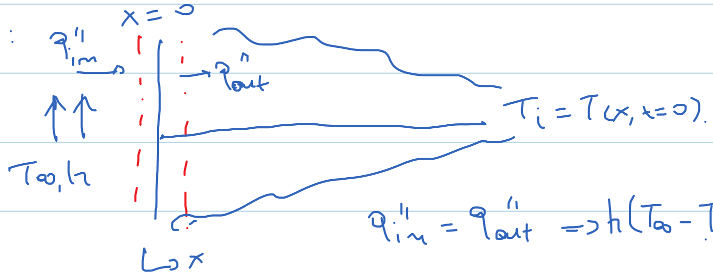
$$T - T_i = \frac{2 q_0'' \left(\frac{\alpha t}{\pi}\right)^{1/2}}{k} \exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_0'' x}{k} \text{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

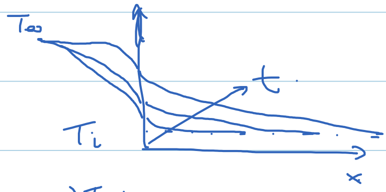
$$q_s'' = -\frac{k}{\sqrt{\pi \alpha t}} (T_i - T_s)$$

$\text{erfc}(x) = 1 - \text{erf}(x).$

$$q_s'' = q''\Big|_{x=0} = q_0''$$

3^{d} Condition:

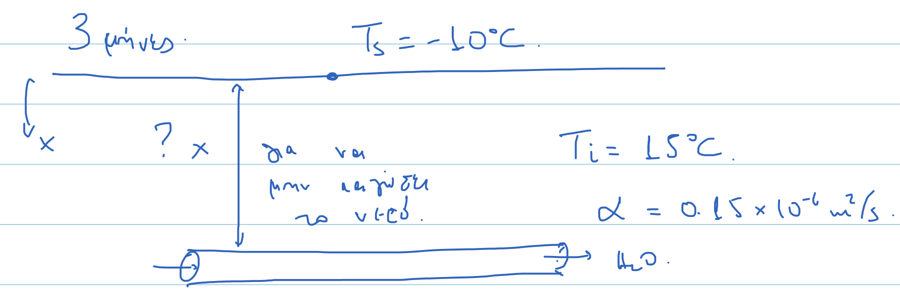




$$q_{in}'' = q_{out}'' \Rightarrow h(T_\infty - T(0,t)) = -k \frac{\partial T}{\partial x}\Big|_{x=0}.$$

$$\frac{T - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\right]\left[\text{erfc}\left(\frac{x}{\sqrt{4\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right].$$

$\text{erfc}(x) = 1 - \text{erf}(x).$



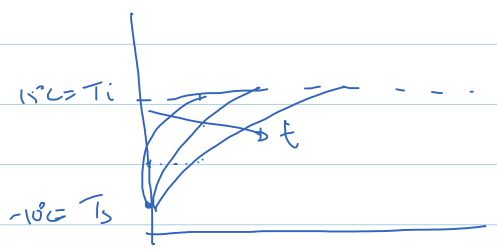
Παραδοχές: - 1D μεταφορά
- Έδαφος ημιάπειρο στερεό.
- Σταθερές ιδιότητες.

Θερμοκρασιακή κατανομή: $\frac{T - T_s}{T_i - T_s} = erf\left(\frac{x}{\sqrt{4\alpha t}}\right) \Rightarrow \frac{0°C - (-10°C)}{15°C - (-10°C)} = erf\left(\frac{x}{\sqrt{4 \cdot 0.15 \times 10^{-6} \frac{m^2}{s} \times (3 \times 30 \times 24 \times 3600) s}}\right)$

$$\frac{10}{25} = erf\left(\frac{x}{\sqrt{4\alpha t}}\right).$$

$$\frac{x}{\sqrt{4\alpha t}} = erfinv\left(\frac{10}{25}\right) = 0.3708. \rightarrow$$

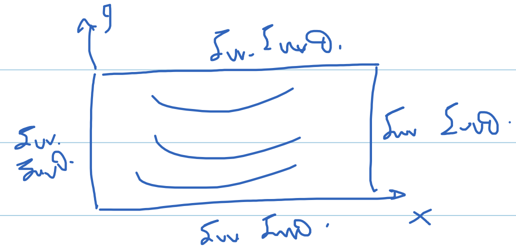
$$\Rightarrow \boxed{x = 0.8\ m.}$$



2D ΜΕΤΑΦΟΡΑ ΘΕΡΜΟΤΗΤΑΣ
ΜΕ ΑΓΩΓΗ ΣΕ ΜΟΝΙΜΗ ΚΑΤΑΣΤΑΣΗ.

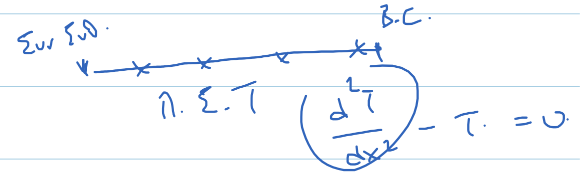
- Αναλυτικές Μέθοδοι Επίλυσης.

  $T(x,y)$ = γνωστή έκφραση . $f(x,y)$



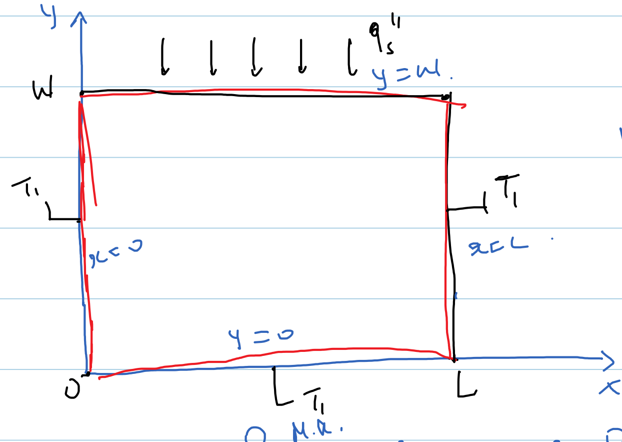
- Προσεγγιστικές -ιι- -ιι-

  (πεπερασμένες διαφορές, πεπερασμένα στοιχεία)



Wednesday, December 16, 2020
9:41 AM

ΑΝΑΛΥΤΙΚΗ ΕΠΙΛΥΣΗ – ΜΕΘΟΔΟΣ ΧΩΡΙΣΜΟΥ ΜΕΤΑΒΛΗΤΩΝ.



Μόνιμη Κατάσταση.

Heat Equation: $\rho c_p \cancel{\frac{\partial T}{\partial t}}^{\,0 \text{ M.K.}} = k\nabla^2 T + \cancel{\dot{q}}^{\,0}$.

$$\Downarrow$$

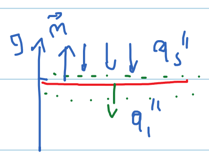
$T = T(x,y)$

$$k\nabla^2 T = 0 \Rightarrow \nabla^2 T = 0 \Rightarrow$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.}$$

@ $x=0,\ y=0,\ x=L$: $T = T_1$.

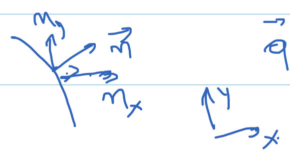
@ $y = W$: $q_s'' = k \left.\frac{\partial T}{\partial y}\right|_{y=W}$.



$q_s'' = q_1''$

$q_1'' = k\frac{\partial T}{\partial y}$.

$$\vec{q}'' = -k\nabla T = -k\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right).$$



$$\vec{q}'' \cdot \vec{n} = -k\frac{\partial T}{\partial x}\cdot n_x - k\frac{\partial T}{\partial y} n_y.$$

Μεταβλητή : $\Theta \equiv T - T_1$.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \Rightarrow \frac{\partial^2}{\partial x^2}(\Theta + T_1) + \frac{\partial^2(\Theta + T_1)}{\partial y^2} = 0. \quad T_1 = ct. \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = 0}$$

@ $x=0$, $y=0$, $x=L$: $T = T_1 \Rightarrow \Theta + T_1 = T_1 \Rightarrow \boxed{\Theta = 0}$

@ $y = W$: $q_s'' = k\left.\frac{\partial T}{\partial y}\right|_{y=W} = k\frac{\partial}{\partial y}(\Theta + T_1) \stackrel{T_1 = ct}{\Longrightarrow} \boxed{k\left.\frac{\partial \Theta}{\partial y}\right|_{y=W} = q_s''}$

Μέθοδος χωρισμού μεταβλητών: $\theta = f(x)\, g(y)$.

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \Rightarrow \frac{\partial^2}{\partial x^2}\left(f(x)\, g(y)\right) + \frac{\partial^2}{\partial y^2}\left(f(x)\, g(y)\right) = 0 \Rightarrow$$

$$\Rightarrow g(y) \cdot \frac{d^2 f}{dx^2} + f(x) \frac{d^2 g}{dy^2} = 0 \Rightarrow$$

$$\Rightarrow \underbrace{-\frac{1}{f(x)} \cdot \frac{d^2 f}{dx^2}}_{F(x)} = \underbrace{\frac{1}{g(y)} \frac{d^2 g}{dy^2}}_{G(y)} = A.$$

$$-\frac{1}{f} \frac{d^2 f}{dx^2} = A.$$

$$\frac{1}{g} \frac{d^2 g}{dy^2} = A.$$

A τι είναι: $A = 0$ ή $A > 0$ ή $A < 0$.

Έστω $A = 0$:

$$\left.\begin{aligned} -\frac{1}{f} \frac{d^2 f}{dx^2} &= 0 \\ \frac{1}{g} \frac{d^2 g}{dy^2} &= 0 \end{aligned}\right] \Rightarrow \left.\begin{aligned} \frac{d^2 f}{dx^2} &= 0 \\ \frac{d^2 g}{dy^2} &= 0 \end{aligned}\right] \Rightarrow \begin{aligned} f &= C_1 \cdot x + C_2 \\ g &= C_3 y + C_4 \end{aligned}$$

$$\left.\begin{aligned} @\, x = 0 &: \theta = 0 \Rightarrow f(x=0) \cdot g(y) = 0 \Rightarrow C_2\, g(y) = 0 \\ x = L &: \theta = 0 \Rightarrow f(x=L)\, g(y) = 0 \Rightarrow (C_1 \cdot L + C_2)\, g(y) = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} &g(y) = 0 \\ &\text{ή} \\ &C_2 = 0 \text{ και } (C_1 L + C_2) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &g(y) = 0 \Rightarrow \theta = 0. \\ &\text{ή} \\ &C_2 = C_1 = 0 \Rightarrow g(x) = 0 \Rightarrow \theta = 0. \end{aligned}$$

$\theta = 0$ δεν ικανοποιεί τη B.C. στο $y = W$.

Αν $A < 0$:

$$\left.-\frac{1}{f} \frac{d^2 f}{dx^2} = A = -\lambda^2.\right\} \Rightarrow \left.\frac{d^2 f}{dx^2} - \lambda^2 f = 0.\right\} \Rightarrow$$

Αν $A<0$:

$$-\frac{1}{f}\frac{d^2f}{dx^2} = A = -\lambda^2 \quad \Rightarrow \quad \frac{d^2f}{dx^2} - \lambda^2 f = 0$$

$$\frac{1}{g}\frac{d^2g}{dy^2} = A = -\lambda^2 \quad \Rightarrow \quad \frac{d^2g}{dy^2} + \lambda^2 g = 0 \quad \Rightarrow$$

$$f = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

$$g = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

@ $x=0$: $\Theta = 0 \Rightarrow f(x=0)\cdot g(y) = 0 \Rightarrow (C_1 + C_2)\, g(y) = 0$

@ $x=L$: $\Theta = 0 \Rightarrow f(x=L)\, g(y) = 0 \Rightarrow (C_1 e^{\lambda L} + C_2 e^{-\lambda L})\, g(y) = 0$

$g(y) \neq 0 \Rightarrow$

$$\left.\begin{aligned} C_1 + C_2 &= 0 \\ C_1 e^{\lambda L} + C_2 e^{-\lambda L} &= 0 \end{aligned}\right\} \Rightarrow C_1 (e^{\lambda L} - e^{-\lambda L}) = 0$$

$\Downarrow$ $C_1 \neq 0$ γιατί αλλιώς $f(x)=0$

$$e^{\lambda L} - e^{-\lambda L} = 0$$

$$\Downarrow$$

$\lambda = 0$ άτοπο.

Άρα $A>0$:

$$-\frac{1}{f}\frac{d^2f}{dx^2} = \lambda^2$$

$$\frac{1}{g}\frac{d^2g}{dy^2} = \lambda^2 \quad \Rightarrow$$

$$\Rightarrow \quad f = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

$$g = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

@ $x=0$: $\Theta = 0 \Rightarrow f(x=0)\cdot g(y) = 0 \Rightarrow C_1 \cdot g(y) = 0 \Rightarrow$
$\Rightarrow C_1 = 0$

@ $x=L$: $\Theta = 0 \Rightarrow f(x=L)\cdot g(y) = 0 \Rightarrow f(x=L) = 0 \Rightarrow$
$\Rightarrow C_2 \sin(\lambda L) = 0 \Rightarrow \sin(\lambda L) = 0 \Rightarrow$

$$\Rightarrow \quad \lambda L = n\pi \quad , \quad n = \cancel{0}, 1, 2, \ldots \Rightarrow$$

$$\Rightarrow \lambda = n\pi / L .$$

@ $y = 0$: $\theta = 0 \Rightarrow f(x) \cdot g(0) = 0 \Rightarrow$

$$\Rightarrow g(0) = 0 \Rightarrow C_3 + C_4 = 0 .$$

$$\theta = f(x)\, g(y) = \left[C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \right] \cdot \left[C_3 \exp(\lambda y) + C_4 \exp(-\lambda y) \right]$$

$$C_1 = 0 \quad , \quad \lambda = n\pi / L \,, \quad n = 1, 2, \ldots$$

$$C_3 = -C_4$$

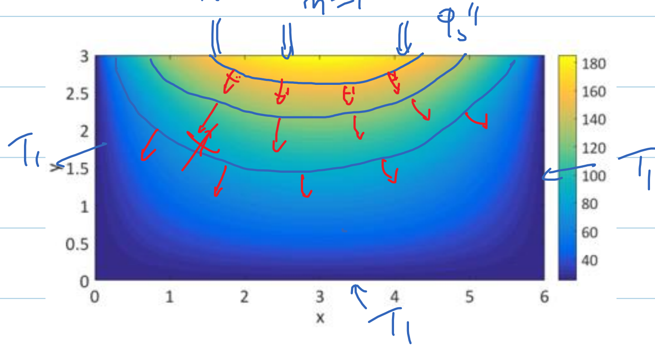
$$\theta = C_2 \sin\left(\frac{n\pi}{L} x\right) 2 C_3 \Big(\underbrace{\frac{\exp\left(\frac{n\pi}{L} y\right) - \exp\left(-\frac{n\pi}{L} y\right)}{2}}_{\sinh\left(\frac{n\pi}{L} y\right)} \Big)$$

$$\theta = C_n' \sin\left(\frac{n\pi}{L} x\right) \sinh\left(\frac{n\pi}{L} y\right).$$

$$\theta = \sum_{n=1}^{\infty} C_n' \sin\left(\frac{n\pi}{L} x\right) \sinh\left(\frac{n\pi}{L} y\right).$$

C_n' υπολογίζω από τη συνθήκη στο $y = W$.

$$T - T_1 = \theta = \frac{2 L q_s''}{k\pi^2} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n^2} \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh\left(\frac{n\pi y}{L}\right)}{\cosh\left(\frac{n\pi W}{L}\right)}$$



$L = 6\,m$

$W = 3\,m$

$k = 0.25\ W/mK.$

$q_s'' = 20\ W/m^2$

$T_1 = 25^\circ C$.